Investigating Black Hole Properties in Modified Gravity Frameworks

Proposed Synopsis for the award of the Ph.D. Degree in Physics

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STATE OF THE ART

One of the best and most extensively tested theories of gravity up till now is general relativity (GR), which Einstein proposed in 1915. Nevertheless, there is still motivation to modify the framework at large scales. One of the main reasons is the discrepancy between the observed dynamics of galaxies and clusters of galaxies and the amount of luminous matter these galaxies and clusters contain [1, 2]. This is usually explained by postulating the existence of exotic dark matter. Neither in laboratory tests nor in satellite missions, any dark matter particle candidate has thus far been observed. An alternative resolution to the problem of the galaxy and galaxy cluster dynamics is a modification of the laws of gravitation on scales where Newtonian gravity or general relativity have not been extensively tested. Among these frameworks, MOG (Modified Gravity) [3] has successfully explained galaxy and galaxy cluster dynamics in the current cosmic epoch without requiring dark matter. Among various modifications of Einstein's general relativity theory, one can construct other gravity-related theories by modifying the GR action. For example, $f(T)$ gravity [4], where T is the torsion scalar in teleparallelism; $f(R)$ gravity [5] with R as the scalar curvature; $f(G)$ gravity with G the Gauss-Bonnet invariant [6]; $f(R,T)$ gravity, where T the trace of the energy-momentum tensor of matter [7]; $f(T,T)$ gravity, where T is the torsion scalar in teleparallelism and T the trace of the energy-momentum tensor of matter [8] etc.

The Yukawa potential in modified gravity, $f(R)$ gravity, and vector-tensor modified gravity theories are prominent models in modified gravity research. Yukawa potential in modified gravity theories can address problems in astrophysics and cosmology without invoking dark matter. $f(R)$ gravity is one of the most studied modified gravity theories. It generalizes Einstein's general relativity by replacing the Ricci scalar R in the Einstein-Hilbert action with a function $f(R)$. Dark energy, cosmic acceleration, gravitational lensing, structure formation, cosmic growth, black hole physics, gravitational waves, Hubble tension, and other cosmological mysteries can be better understood with this modification. In addition to the metric tensor of general relativity (GR), vector-tensor theories of gravity also introduce a vector field. Vector- tensor modified gravity has application in explaining the dark energy and cosmic acceleration, dark matter alternative, gravitational wave, gravitational lensing, black hole physics etc.

The characteristics of circular orbits surrounding black holes diverge from those predicted by Newtonian potential. Unstable circular orbits manifest near a black hole for time-like geodesics. Circular orbits are also established for null geodesics. These circular orbits around black holes are associated with significant gravitational phenomena, including a black hole's shadow and an accretion disk. The dynamics of particles, whether massless or large, neutral or charged, in proximity to a black hole is one of the most compelling issues in black hole astrophysics. Numerous theories have been proposed concerning geodesic motion in the vicinity of black holes. Chandrasekhar [9] conducted a thorough examination of the geodesic structure of black holes, including Schwarzschild, Reissner-Nordström, and Kerr types. He also examined the dynamics of test particles by employing the Newman-Penrose formalism. The magnetic field phenomenon causes an attraction of black holes within their accretion disks. The magnetic field surrounding the black hole is generated by plasma [10]. The magnetic field influences both plasma and test particles, although it does not alter the geometric structure of black holes [11]. Magnetic fields facilitate energy transmission to particles circling near the black hole [12]. A multitude of parameters influencing particle motion around Schwarzschild and Reissner-Nordström black holes have been analyzed, with a comprehensive study detailed in [13, 14]. Comprehensive investigations of neutral particle motion [15] and the dynamics of charged particles [16] in Reissner-Nordström spacetime have been conducted. The geodesics of electrically and magnetically charged test particles in the Reissner-Nordström spacetime have been examined in [17]. The capture cross-section of massless and heavy particles by the charged black hole has been examined in [18]. The innermost stable circular orbits of charged, rotating test particles have been examined in [19]. [20] examines the acceleration of charged particles by black holes in Reissner-Nordström spacetime. The importance of the electric and magnetic charge of astrophysical black holes has been addressed in [21]. The dynamical motion of a charged particle near a revolving black hole in the presence of an external magnetic field has been examined in. [22]. Black holes possess a temperature and undergo evaporation as a result of Hawking radiation. The entropy of black holes is generalized to Wald entropy [23]. The entropy formula is applicable for calculating the entropy of a black hole inside any modified gravitational theory. Black hole entropy has been examined in the contexts of Gauss-Bonnet gravity [24] and Lovelock gravity [25]. Research on black hole thermodynamics has also been conducted within the framework of massive gravity.

Rayimbaev [26] has investigated the characteristics of the MOG field and the motion of test particles in the vicinity of normal black holes (BHs).

Recent advancements in cosmology have focused on observational investigations of dark energy and its possible explanations within modified gravity frameworks. Utilizing various datasets, researchers have explored deviations from the standard ΛCDM model [27]. Observational constraints on modified gravity models, have shown promising results in explaining cosmic acceleration [28]. These developments underscore the growing synergy between theoretical frameworks and observational datasets, enabling more robust testing of alternative explanations for the universe's accelerated expansion.

PROBLEMS TO BE INVESTIGATED

1. Black Hole solutions in modified gravity

2. Thermodynamic properties of black holes in modified gravity

3. Charged particles and QPOs (Quasi periodic oscillations) around charged black holes in modified gravity

4. Gravitational Lensing

5. Dark energy in modified gravity models

OBJECTIVES

The study will initially focus on one of the following objectives. Upon achieving successful outcomes, efforts will be made towards pursuing subsequent objectives.

1. **To explore black hole solutions in modified gravity frameworks**:
    - Derive exact and approximate metrics for black holes in f(R) gravity, Scalar-Tensor-Vector Gravity (STVG), and Yukawa-modified gravity.

- Analyse how modifications to general relativity (GR) affect spacetime geometry, event horizons, and singularities.

2. **To investigate the thermodynamic properties of black holes in modified gravity**:
    - Study entropy, temperature, and phase transitions in the context of corrections introduced by modified gravity theories.
    - Examine stability criteria for black holes by analyzing heat capacity and free energy.
3. **To analyse quasi-periodic oscillations (QPOs) in accretion disks near black holes**.
4. **To examine gravitational lensing effects in modified gravity:**
    - Derive and simulate deflection angles for light rays near black holes and massive objects.
    - Explore how lensing patterns, Einstein rings, and black hole shadows are altered by modifications to GR
5. **Observational analysis of dark energy in modified gravity models and its implications for black holes.:**
    - Investigating observational datasets such as H(z), Type Ia supernovae, BAO, and CMB to understand the impact of modified gravity on dark energy.

HYPOTHESIS

Primary Hypothesis: Black holes, as solutions to modified gravity theories such as f(R) gravity, scalar-tensor-vector gravity (STVG), and Yukawa-modified gravity, exhibit observable deviations in their thermodynamic properties, particle dynamics, which can be used to test the validity of these alternative theories and constrain their parameters

Secondary Hypotheses

1. **Thermodynamics**: Modified gravity theories introduce corrections to the entropy, temperature, and phase transitions of black holes. These corrections can provide insights into quantum gravitational effects and deviations from classical GR predictions.

2. **Quasi-Periodic Oscillations (QPOs)**: Charged black holes in modified gravity influence the motion of charged particles and the frequencies of QPOs. These frequencies provide a unique signature to distinguish GR from modified gravity models.
3. **Yukawa Potential**: The addition of a Yukawa-like term to the gravitational potential modifies galaxy rotation curves, black hole shadow sizes, and light deflection patterns, mimicking dark matter effects without invoking exotic particle.

RESEARCH DESIGN AND METHODOLOGY:

A. Black holes and thermodynamics of black holes
B. Modified gravity theories
C. Quasi periodic oscillations (QPOs) in black holes
D. Dark energy in Modified Gravity models

A. Black holes and thermodynamics of black holes :

Black holes: The search for black hole candidates began in the 1970s, leading to strong observational evidence for two classes: stellar-mass black holes $\left(M \approx 3-100 M_{\oplus}\right)$ and supermassive $\left(M \approx 10^{5}-10^{10} M_{\oplus}\right)$black holes. A third category, intermediate-mass black holes, likely exists, but their properties remain debated, and precise mass measurements are lacking. Black holes arise as solutions to Einstein's equations, where gravity's non-linearity is most pronounced. Schwarzschild discovered the first static solution in 1916, followed by charged generalizations from Reissner (1916) and Nordström (1918). Kerr later identified a family of rotating black hole solutions in 1963, with Newman et al. (1965) extending them to charged Kerr-Newman solutions. These charged Kerr (Kerr-Newman) solutions form a three-parameter family, whose space time metric is given by:

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\left(r^2 + a^2 - \Delta\right)}{\Sigma}dtd\phi +$$
$$\left[\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

where $\Sigma = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 + a^2 + e^2 - 2Mr$ and e, α and M are the three parameters of the family. When $e = 0,$ the spacetime metric reduces to the vacuum Kerr metric. When $a = 0,$ we recover the Reissner- Nordstrom solutions, and when $e = a = 0,$ the above metric reduces to Schwarzschild solution. It follows that the three-parameter family contains all known solutions for stationary black holes, and that the Kerr-Newman metric is the most universal solution for spacetime geometry. Uncharged black holes are described by the Schwarzschild and Kerr metrics, whereas electric charge is taken into consideration by the Reissner-Nordstrom and Kerr-Newman metrics. Metrics for non-spinning black holes are the Schwarzschild and Reissner-Nordstrom metrics, while measurements for rotating black holes are the Kerr and Kerr-Newman metrics.

Thermodynamics of black holes: Investigating the thermodynamics of black holes reveals a fascinating link between quantum mechanics, general relativity, and thermodynamics. It provides a framework to understand how black holes behave as thermodynamic systems. There are four laws that black holes follow, much like in classical thermodynamics. Thermodynamics of black holes can be studied in modified gravity. It should be emphasized that black holes must have an entropy for the second law of thermodynamics to not be violated.

Table 1: Mathematical analogy between laws of black holes and thermodynamics

Law	Thermodynamics	Black holes
Zeroth	T constant throughout body in thermal equilibrium	k constant over horizon of stationary black hole satisfy an important property analogous to the property of temperature.
First	$dE = Tds +$ work terms	$dM = \frac{1}{8\pi} k\, dA + \Omega_H dJ$
Second	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $k = 0$ by a physical process

Black holes emit radiation due to quantum effects near the event horizon, called Hawking radiation. Hawking radiation emitted by the black hole was thermal and given as

$$T = \frac{1}{2\pi}\frac{\kappa\hbar}{k_B c}$$

Where T is proportional to the black hole's surface gravity κ, k_B is Boltzmann's constant, c is the speed of light and $\hbar$ is the Planck's constant. The exact constant of proportionality that relates the entropy to the horizon surface area

$$S = \frac{k_B}{4}\frac{A}{\ell_p^2}$$

Where ℓ_p is the Planck's length. In units where $\hbar = c = k_B = 1$, the temperature is given by,

$$T = \frac{\kappa}{2\pi}$$

while entropy is given by $S = A/4$.

In modified gravity ($f(R)$ gravity), black hole temperature and entropy are modified, and are expressed as:

$$T = \frac{k}{2\pi},\ \ k = \frac{1}{2}\frac{\partial f(R)}{\partial r}\bigg|_{r=r_H}$$

$$S = \frac{k_B}{4G}\frac{A}{f'(R)}$$

B. Modified gravity (Ex. $f(R)$ gravity, STVG gravity and Yukawa gravity) theories:

These theories come about by a straightforward generalization of the Lagrangian in the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \qquad (1)$$

where $\kappa = 8\pi G$, G is the gravitational constant, g is the determinant of the metric and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, and the Ricci-tensor is defined as $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$. Here we use the metric signature $(+,-,-,-)$ and $(c = \hbar = 1)$. We have

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, f(R),$$

Here $f(R)$ can be represented by series expansion as,

$$f(R) = ... + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + ...$$

where the α_i and β_i coefficients have the appropriate dimensions and Λ is the cosmological constant. Beginning from the action and adding a matter term S_M , the total action for $f(R)$ gravity takes the form

$$S_{met} = \frac{1}{2K}\int d^4x\sqrt{-g}\, f(R) + S_M(g_{\mu\nu}, \psi)\ ,$$

where ψ collectively denotes the matter fields. Variation with respect to the metric gives, after some manipulations, gives field equation of $f(R)$.

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\,\Box\right]f'(R) = kT_{\mu\nu},$$

$$\text{where } T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$

Here, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric, and $\Box = \nabla^\mu\nabla_\mu$. The upto date modified gravity (MOG) is a fully relativistic theory of gravitation that is derived from a relativistic action principle involving scalar, tensor, and vector fields. MOG has evolved as a result of investigations of non-symmetric gravity theory and, most recently, it has taken the form of Scalar-Tensor-Vector Gravity (STVG), that includes the Einstein-Hilbert action and extend it using new fields actions and new constants. The lagrangian

density in the STVG (Scalar-Tensor-Vector-Gravity) formulation of MOG can be written as $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_\phi + \mathcal{L}_S + \mathcal{L}_M$. Here, $\mathcal{L}_G$ is the lagrangian density of original Einstein-Hilbert gravity action, $\mathcal{L}_\phi$is the lagrangian density of a massive vector field, $\mathcal{L}_S$ is the lagrangian density of scalar fields and $\mathcal{L}_M$ is the lagrangian density for pressure-less matter.

The STVG action (similar to (1)) integral takes the form

$$S = \int \mathcal{L} dx^4$$

The field equations of STVG theory have been developed by John Moffat from the action integral using the variational principle. Gravitational lensing describes how the path of null waves originating from a distant source is altered as they pass near a massive object. This change in direction results from the gravitational field of the massive lens, which bends the path of these waves in line with GR. Time delay in modified gravity and the effect of modified gravity in lensing of fast radio bursts have opened up a new window in astronomy.

Yukawa Potential in Modified gravity

Yukawa potential arises in modified gravity theories such as scalar tensor gravity, Massive gravity theories and generalized $f(R)$ gravity. The Yukawa potential modifies the classical Newtonian gravitational potential by introducing a short-range, exponentially decaying correction. It has the form

$$\phi = -\frac{GM}{r}\left(1 + \alpha e^{-r/\lambda}\right),$$

Where $-GM/r$ is the Newtonian gravitational potential α (a dimensionless constant) is the strength of the Yukawa correction, λ is the range of the Yukawa interaction(characteristic length scale) and r is the radial distance from the massive object. The Yukawa correction $\propto e^{-r/\lambda}$ is a short-range force that weakens as r increases, with an exponential decay governed by λ. The modified gravitational potential in STVG includes an extra term from the vector field.

$$\phi = -\frac{GM}{r}\left(1 + \alpha e^{-\mu r}\right)$$

Here, α is the strength of the modified gravitational interaction, μ is the inverse range of the vector field, and G is the gravitational constant.

C. Quasi periodic oscillations (QPOs) in black holes

The dynamics of charged particles in accretion disks play a crucial role in forming quasi-periodic oscillations (QPOs). Extracting QPO frequencies from X-ray data provides insights into both the black hole and the disk. X-ray binary systems, formed through various mechanisms, consist of a black hole or neutron star with a companion star that transfers material, creating a stable accretion disk. Strong gravitational forces near the black hole horizon generate high-frequency QPOs, influencing the disk structure. QPO formation is linked to factors such as Keplerian motion, black hole spin, and alternative gravitational theories. Understanding the disk's dynamic structure and oscillations is essential for determining black hole properties like mass and spin.

Observations across different electromagnetic frequencies, particularly X-rays, help characterize disk oscillations and gravitational strength. In Modified Gravity (MOG) models, perturbing the disk naturally leads to QPOs due to the interaction between the modified gravitational field and disk oscillations. The QPO frequencies $\nu_U = f(M.a,\alpha)$ and $\nu_L = g(M.a,\alpha)$, where f and g are functions depending on black hole mass, spin, and MOG coupling parameter α. A comprehensive stability analysis requires considering various physical parameters that govern disk dynamics and perturbations. To analyse QPO frequencies, particle motion in the black hole's effective potential is examined through three characteristic frequencies: orbital (Keplerian), radial epicyclic, and vertical epicyclic.

Table2: Types of frequencies

Orbital Frequency (ν_ϕ)	Frequency at which a particle completes circular orbit	$\nu_\phi = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}}$
Radial Epicyclic Frequency (ν_r)	Frequency of small radial oscillations around a stable orbit	$\nu_r^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^3\frac{d\Phi_{eff}}{dr}\right)$
Vertical Epicyclic Frequency(ν_θ)	Frequency of oscillations perpendicular to accretion disk	$\nu_\theta^2 = \frac{\Phi_{eff}}{r^3}$

Resonant oscillations may be linked to the innermost stable circular orbit (ISCO), offering insights into strong gravity and testing general relativity. The presence of QPOs in accretion disks supports this idea. The ISCO marks the region where gravity is extreme and accretion disk physics is most complex. Beyond this point, matter spirals inward, inevitably falling into the black hole. Circular orbits can be stable when the effective potential satisfies the condition $\partial_{rr} V_{eff}\left(r_{ISCO}\right) > 0$, while ISCO satisfies $\partial_{rr} V_{eff}\left(r_{ISCO}\right) = 0$, mostly together with the conditions $V_{eff} = \varepsilon$ and $V'_{eff} = 0$.

D. Dark Energy in modified gravity models:

Recent cosmological studies explore the link between dark energy (DE) and modified gravity (MOG), with the Hubble parameter H(z) as a key observational tool. The universe's evolution follows a modified Friedmann equation incorporating MOG terms:

$$H^2(z) = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_{de} \left(\frac{1+z}{a(z)}\right)^3\right]$$

Where H_0 is the Hubble constant, Ω_m represents the matter density parameter, and Ω_{de} is the dark energy parameter. Observations from H(z), Type Ia supernovae, BAO, and CMB tests compare Λ-CDM with MOG models, which propose alternative explanations for cosmic acceleration. Signatures like gravitational lensing, black hole shadows, and QPOs help probe these theories, with black hole dynamics offering crucial insights into MOG and dark energy interactions.

TOOLS TO BE USED

1. **Theoretical and Analytical Models**:
    - Derivation of black hole solutions and their thermodynamic properties within modified gravity theories.
    - Analytical study of Yukawa potentials and their effects on particle dynamics and spacetime curvature.
2. **Numerical Simulations**:
    - Simulating black hole metrics in f(R) Yukawa-modified, and STVG frameworks.
    - Numerical solutions to geodesic equations for particle motion and light propagation.

3. **Computational Tools**:

   - **Maple/Mathematica**: For symbolic computation of field equations and geodesics.
   - **Python/MATLAB**: For numerical simulations of particle motion and light trajectories.

SIGNIFICANCE OF THE STUDY

1. **Theoretical Importance**:

   - Challenges the dominance of GR by offering alternative explanations for phenomena traditionally attributed to dark matter and dark energy.
   - Provides a deeper connection between quantum mechanics, thermodynamics, and gravitational physics.

2. **Observational Importance**:

   - Uses cutting-edge observational data (e.g., black hole shadows) to test predictions of modified gravity.

CONTRIBUTION TO EXISTING KNOWLEDGE

This study will:

1. Extend the understanding of black hole thermodynamics in modified gravity frameworks, addressing entropy and temperature corrections.
2. Provide new insights into QPOs near charged black holes, revealing how modified gravity influences accretion disk dynamics.
3. Bridge the gap between theoretical predictions and observational data, particularly in the context of black hole shadows and QPOs

FUTURE WORK

Future research could extend the study of black hole solutions to additional modified gravity such as Horndeski or Gauss-Bonnet gravity models and employ numerical simulations to examine black hole behaviour under extreme conditions. Gravitational wave observations could also be used to test predictions, while further analysis of the thermodynamic properties and stability of exotic black holes may provide deeper insights into gravity in modified frameworks.

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